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Generalization of multifractal theory within quantum calculus

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Abstract – On the basis of the deformed series in quantum calculus, we generalize the partition function and the mass exponent of a multifractal, as well as the average of a random variable distributed over a self-similar set. For the partition function, such expansion is shown to be determined by binomial-type combinations of the Tsallis entropies related to manifold deformations, while the mass exponent expansion generalizes the known relation $\tau_q = D_q(q-1)$. We find the equation for the set of averages related to ordinary, escort, and generalized probabilities in terms of the deformed expansion as well. Multifractals related to the Cantor binomial set, exchange currency series, and porous-surface condensates are considered as examples.

Introduction. – Fractal conception [1] has become a widespread idea in contemporary science (see refs. [2–4] for a review). Self-similarity is known to be a characteristic feature of fractal sets: if one takes a part of the whole set, it looks like the original set after appropriate scaling. The formal basis of self-similarity is the power law function $F \sim \ell^h$ with the Hurst exponent h (for time series, the value F is reduced to the fluctuation amplitude and ℓ is the interval size within which this amplitude is determined). While the simple case of monofractal is characterized by a single exponent h, a multifractal system is described by a continuous spectrum of exponents, the singularity spectrum $h(q)$ with argument q being the exponent deforming the measures of the elementary boxes that cover the fractal set [5]. On the other hand, the parameter q represents a self-similarity degree of a homogeneous function being intrinsic in self-similar systems [6] (in this way, within non-extensive thermostatistics, this exponent expresses the escort probability $P_i \propto p_i^q$ in terms of the original one p_i [7,8]). In physical applications, a key role is played by the partition function $Z_q \sim \ell^{\tau(q)}$ with ℓ as the characteristic size of the boxes covering the multifractal and the exponent $\tau(q)$ connected with the generalized Hurst exponent $h(q)$ by the relation $\tau(q) = qh(q) - 1$.

As fractals are scale-invariant sets, it is natural to apply the quantum calculus to describe multifractals. Indeed, quantum analysis is based on the Jackson derivative

$$\mathcal{D}_x^\lambda = \frac{\lambda^{x\partial_x} - 1}{(\lambda - 1)x}, \qquad \partial_x \equiv \frac{\partial}{\partial x} \tag{1}$$

that yields the variation of a function $f(x)$ with respect to the scaling deformation λ of its argument [9,10]. First, this idea has been realized in the work [6] where the support space of the multifractal has been proposed to deform by means of the action of the Jackson derivative (1) on the variable x reduced to the size ℓ of the covering boxes. In this letter, we use a quite different approach wherein deformation is applied to the multifractal parameter q itself to vary it by means of the finite dilatation $(\lambda - 1)q$ instead of the infinitesimal shift dq. We demonstrate below that the related description allows one to generalize the definitions of the partition function, the mass exponent, and the averages of random variables on the basis of the deformed expansion in power series over the difference $q - 1$. We apply the proposed formalism to the consideration of multifractals in mathematical physics (the Cantor binomial set), econophysics (exchange currency series), and solid-state physics (porous-surface condensates).

Formulation of the problem. – Following the standard scheme [3,5], we consider a multifractal set covered by the elementary boxes $i = 1, 2, \ldots, W$ with $W \to \infty$. Its properties are known to be determined by the partition

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function

$$Z_q = \sum_{i=1}^{W} p_i^q, \tag{2}$$

that takes the value $Z_q = 1$ at $q = 1$, in accordance with the normalization condition. Since $p_i \leqslant 1$ for all boxes i, the function (2) decreases monotonically from maximum magnitude $Z_q = W$ related to $q = 0$ to extreme values $Z_q \simeq p_{\rm ext}^q$ which are determined in the $|q| \to \infty$ limit by the maximum probability $p_{\max}$ on the positive half-axis $q > 0$ and by the minimum magnitude $p_{\min}$ on the negative one. In the simplest case of the uniform distribution $p_i = 1/W$, fixed by the statistical weight $W \gg 1$, one has the exponential decay $Z_q = W^{1-q}$.

The cornerstone of our approach is a generalization of the partition function (2) by means of introducing a deformation parameter λ which defines, together with the self-similarity degree q, *a modified partition function* $\mathcal{Z}_q^\lambda$ reduced to the standard form Z_q at $\lambda = 1$. To find the explicit form of the function $\mathcal{Z}_q^\lambda$, we expand the difference $\mathcal{Z}_q^\lambda - Z_\lambda$ into the deformed series in powers of the difference $q - 1$:

$$\mathcal{Z}_q^\lambda := Z_\lambda - \sum_{n=1}^{\infty} \frac{\mathcal{S}_\lambda^{(n)}}{[n]_\lambda!}(q-1)_\lambda^{(n)}, \qquad Z_\lambda = \sum_{i=1}^{W} p_i^\lambda. \tag{3}$$

For arbitrary x and a, the deformed binomial [9,10]

$$\begin{aligned}(x+a)_\lambda^{(n)} &= (x+a)(x+\lambda a)\dots(x+\lambda^{n-1}a) \\ &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_\lambda \lambda^{\frac{m(m-1)}{2}} x^m a^{n-m}, \ n \geqslant 1 \end{aligned} \tag{4}$$

is determined by the coefficients $\left[{n \atop m}\right]_\lambda = \frac{[n]_\lambda!}{[m]_\lambda![n-m]_\lambda!}$, where generalized factorials $[n]_\lambda! = [1]_\lambda [2]_\lambda \dots [n]_\lambda$ are given by the basic deformed numbers

$$[n]_\lambda = \frac{\lambda^n - 1}{\lambda - 1}. \tag{5}$$

The coefficients of the expansion (3)

$$\mathcal{S}_\lambda^{(n)} = -\left(q\mathcal{D}_q^\lambda\right)^n Z_q\big|_{q=1}, \qquad n \geqslant 1 \tag{6}$$

are defined by the n-fold action of the Jackson derivative (1) on the original partition function (2).

Generalized entropies. – Simple calculations give the explicit expression

$$\mathcal{S}_\lambda^{(n)} = -\frac{[Z_\lambda - 1]^{(n)}}{(\lambda-1)^n}, \qquad n \geqslant 1. \tag{7}$$

Hereafter, we use *the functional binomial*

$$[x_t + a]^{(n)} := \sum_{m=0}^{n} \binom{n}{m} x_{t^m} a^{n-m}, \tag{8}$$

defined with the standard binomial coefficients $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ for an arbitrary function $x_t = x(t)$ and a constant a. The definition (8) is obviously reduced to the Newton binomial for the trivial function $x_t = t$. The most crucial difference of the functional binomial from the ordinary one is displayed at $a = -1$ in the limit $t \to 0$, when all terms of the sum (8), apart from the first, $x_{t^0} = x_1$, are proportional to $x_{t^m} \to x_0$ to give

$$\lim_{t\to 0} [x_t - 1]^{(n)} = (-1)^n (x_1 - x_0). \tag{9}$$

At $t = 1$, one has $[x_1 - 1]^{(n)} = 0$.

It is easy to see that the set of coefficients (7) is expressed in terms of the Tsallis entropy [7]

$$S_\lambda = -\sum_i \ln_\lambda(p_i) p_i^\lambda = -\frac{Z_\lambda - 1}{\lambda - 1}, \tag{10}$$

where the generalized logarithm $\ln_\lambda(x) = \frac{x^{1-\lambda}-1}{1-\lambda}$ is used. As the λ deformation grows, this entropy decreases monotonically taking the Boltzmann-Gibbs form $S_1 = -\sum_i p_i \ln(p_i)$ at $\lambda = 1$. The obvious equality

$$\mathcal{S}_\lambda^{(n)} = -\frac{[(1-\lambda)S_\lambda]^{(n)}}{(\lambda-1)^n}, \qquad n \geqslant 1 \tag{11}$$

expresses in explicit form the entropy coefficients (7) in terms of the Tsallis entropy (10) that relates to manifold deformations λ^m, $0 \leqslant m \leqslant n$. At $\lambda = 0$ when $Z_0 = W$, the limit (9) gives $[S_0]^{(n)} = [Z_0 - 1]^{(n)} = (-1)^{n-1}(W-1)$, so that $\mathcal{S}_0^{(n)} = W - 1 \simeq W$. Respectively, in the limit $\lambda \to 1$ where $S_\lambda \to S_1$ and $[(1-\lambda)S_\lambda]^{(n)} \to (1-\lambda)^n S_1^n$, one obtains the sign-changing values $\mathcal{S}_1^{(n)} \to (-1)^{n-1} S_1^n$. Finally, the limit $|\lambda| \to \infty$ where $S_\lambda \sim \lambda^{-1}$ and $[(1-\lambda)S_\lambda]^{(n)} \sim (-1)^n$ is characterized by the sign-changing power asymptotics $\mathcal{S}_\lambda^{(n)} \sim (-1)^{n-1}\lambda^{-n}$.

For the uniform distribution when $Z_\lambda = W^{1-\lambda}$, the dependence (10) is characterized by the value $S_0 \simeq W$ in the limit $\lambda \ll 1$ and the asymptotics $S_\lambda \sim 1/\lambda$ at $\lambda \gg 1$ (in the point $\lambda = 1$, one obtains the Boltzmann entropy $S_1 = \ln(W)$). As a result, with the λ growth along the positive half-axis, the coefficients (7) decrease from the magnitude $\mathcal{S}_0^{(n)} \simeq W$ to the sign-changing values $\mathcal{S}_1^{(n)} = (-1)^{n-1}[\ln(W)]^n$ and then tend to the asymptotics $\mathcal{S}_\lambda^{(n)} \sim (-1)^{n-1}\lambda^{-n}$; with the $|\lambda|$ growth along the negative half-axis, the coefficients (7) vary non-monotonically tending to $\mathcal{S}_\lambda^{(n)} \sim -|\lambda|^{-n}$ at $\lambda \to -\infty$.

Generalized fractal dimensions. – Within a pseudo-thermodynamic picture of multifractal sets [8], effective values of the free energy τ_q, the internal energy α, and the entropy f are defined as follows:

$$\tau_q = \frac{\ln(Z_q)}{\ln(\ell)}, \quad \alpha = \frac{\sum_i P_i \ln p_i}{\ln(\ell)}, \quad f = \frac{\sum_i P_i \ln P_i}{\ln(\ell)}. \tag{12}$$

Here, $\ell \ll 1$ stands for a scale, p_i and P_i are original and escort probabilities connected with the definition

$$P_i(q) = \frac{p_i^q}{\sum_i p_i^q} = \frac{p_i^q}{Z_q}. \tag{13}$$

Inserting the last equation into the second expression (12), one obtains the Legendre transform $\tau_q = q\alpha_q - f(\alpha_q)$, where q plays the role of the inverse temperature and the internal energy is specified with the state equation $\alpha_q = \frac{\mathrm{d}\tau_q}{\mathrm{d}q}$ [3]. It is easy to convince oneself that the escort probability (13) is generated by the mass exponent given by the first definition (12) by taking eq. (2) into account:

$$qP_i(q) = \ln(\ell)p_i\frac{\partial\tau_q}{\partial p_i} = \frac{\partial\ln(Z_q)}{\partial\ln(p_i)}. \quad (14)$$

Along the line of the generalization proposed, we introduce further *a deformed mass exponent* τ_q^λ related to the original one τ_q according to the condition $\tau_q = \lim_{\lambda\to1}\tau_q^\lambda$. By analogy with eq. (3), we expand this function into the deformed series

$$\tau_q^\lambda := \sum_{n=1}^{\infty}\frac{D_\lambda^{(n)}}{[n]_\lambda!}(q-1)_\lambda^{(n)}, \quad (15)$$

which is a generalization of the known relation $\tau_q = D_q(q-1)$ connecting the mass exponent τ_q with the multifractal dimension spectrum D_q [3]. Similarly to eqs. (6), (7), the coefficients $D_\lambda^{(n)}$ are expressed in the form

$$D_\lambda^{(n)} = \left(q\mathcal{D}_q^\lambda\right)^n\tau_q\big|_{q=1} = \frac{[\tau_\lambda - 1]^{(n)}}{(\lambda-1)^n}, \quad n\geqslant 1, \quad (16)$$

where the use of the definition (8) implies that the term with $m=0$ should be suppressed because $\tau_{\lambda^0}=0$. At $n=1$, the last equation (16) is obviously reduced to the ordinary form $D_\lambda^{(1)} = \tau_\lambda/(\lambda-1)$, while the coefficients $D_\lambda^{(n)}$ with $n>1$ include terms proportional to τ_{λ^m} to be related to the manifold deformations λ^m, $1<m\leqslant n$. To this end, the definition (16) yields a hierarchy of the multifractal dimension spectra related to multiplying deformations of different powers n.

Making use of the limit (9), where the role of the function x_t is played by the mass exponent τ_λ with $\tau_0=-1$ and $\tau_1=0$, gives the value $[\tau_0-1]^{(n)}=(-1)^n$ at the point $\lambda=0$, where the coefficients (16) take the value $D_0^{(n)}=1$ related to the dimension of the support segment. In the limits $\lambda\to\pm\infty$, the behavior of the mass exponent $\tau_\lambda\simeq D_{\rm ext}\lambda$ is determined by extreme values, $D_{\rm ext}$, of the multifractal dimensions which are reduced to the minimum value $D_{\min}=D_\infty^{(n)}$ and the maximum one $D_{\max}=D_{-\infty}^{(n)}$ [3]. On the other hand, in the limits $\lambda\to\pm\infty$, the extreme values $Z_\lambda\simeq p_{\rm ext}^\lambda$ of the partition function (2) are determined by the related probabilities $p_{\rm ext}$. As a result, the first definition (12) gives the mass exponents $\tau_\lambda\simeq\lambda\frac{\ln(p_{\rm ext})}{\ln(\ell)}$ and the coefficients (16) tend to the minimum value $D_\infty^{(n)}\simeq\frac{\ln(p_{\max})}{\ln(\ell)}$ at $\lambda\to\infty$ and to the maximum one $D_{-\infty}^{(n)}\simeq\frac{\ln(p_{\min})}{\ln(\ell)}$ at $\lambda\to-\infty$.

For the uniform distribution whose partition function is $Z_\lambda=W^{1-\lambda}$, the expression $Z_\lambda:=\ell^{\tau_\lambda}$ gives the fractal dimension $D=\frac{\ln(W)}{\ln(1/\ell)}$ which tends to $D=1$ when the size of the covering boxes ℓ tends to the inverse statistical weight $1/W$. Being unique, this dimension relates to a monofractal with the mass exponent $\tau_\lambda=D(\lambda-1)$ whose insertion into the definition (16) yields the equal coefficients $D_\lambda^{(n)}=D$ for all orders $n\geqslant1$.

Relations between generalized entropies and fractal dimensions. – Since either of the deformed series (3) and (15) describes a multifractal completely, their coefficients should be connected in some way. It is easy to find an explicit relation between the first of these coefficients $S_\lambda=\ln_\lambda\left(Z_\lambda^{\frac{1}{1-\lambda}}\right)$ and $D_\lambda=\frac{\ln(Z_\lambda)}{(\lambda-1)\ln(\ell)}$, being the Tsallis entropy $S_\lambda=\mathcal{S}_\lambda^{(1)}$ and the multifractal dimension $D_\lambda=D_\lambda^{(1)}$. The use of the relation $Z_\lambda=\ell^{\tau_\lambda}$, the connection $\tau_\lambda=D_\lambda(\lambda-1)$, and the Tsallis exponential $\exp_\lambda(x)=[1+(1-\lambda)x]^{\frac{1}{1-\lambda}}$ yields the expressions

$$S_\lambda=\ln_\lambda(W^{D_\lambda}),\qquad D_\lambda=\frac{\ln\left[\exp_\lambda(S_\lambda)\right]}{\ln(W)}, \quad (17)$$

where the statistical weight $W=1/\ell$ is used. Unfortunately, it is impossible to set any closed relation between the coefficients (7) and (16) at $n>1$. However, the use of the partition function $Z_\lambda=W^{-D_\lambda(\lambda-1)}$ allows us to write

$$D_\lambda^{(n)}=\frac{[D_\lambda(\lambda-1)-1]^{(n)}}{(\lambda-1)^n}, \quad (18)$$

$$\mathcal{S}_\lambda^{(n)}=-\frac{\left[W^{-D_\lambda(\lambda-1)}-1\right]^{(n)}}{(\lambda-1)^n}. \quad (19)$$

Thus, knowing the first coefficients of expansions (3) and (15) connected with the relations (17), one can obtain them for arbitrary orders $n>1$.

Random variable distributed over a multifractal set. – Let us consider an observable ϕ_i distributed over a multifractal set with the average $\langle\phi\rangle_q^\lambda=\sum_i\phi_i\mathcal{P}_i(q,\lambda)$. The related probability is determined by the equation

$$\lambda\mathcal{P}_i(q,\lambda):=p_i+\ln(\ell)p_i\frac{\partial\tau_q^\lambda}{\partial p_i}, \quad (20)$$

that generalizes eq. (14) for the escort probability due to the λ deformation. Taking into account eqs. (14)–(16), this average can be expressed in terms of the deformed series

$$\lambda\langle\phi\rangle_q^\lambda=\langle\phi\rangle+\sum_{n=1}^{\infty}\frac{[\lambda\langle\phi\rangle_\lambda-1]^{(n)}}{[n]_\lambda!(\lambda-1)^n}(q-1)_\lambda^{(n)}, \quad (21)$$

where $\langle\phi\rangle=\sum_i\phi_ip_i$ and $\langle\phi\rangle_\lambda=\sum_i\phi_iP_i(\lambda)$. This equation allows one to find the mean value $\langle\phi\rangle_q^\lambda$ *vs.* the self-similarity degree q at fixed λ deformation. In the case $\lambda=q$ when definition (4) gives $(q-1)_q^{(n)}=0$ for $n>1$, eq. (21) yields the connection $\langle\phi\rangle_q^q=\langle\phi\rangle_q$ between the mean values related to the generalized and escort probabilities given by eqs. (20) and (14), respectively. At the point $\lambda=0$, where, according to eqs. (4), (5), (9) $(q-1)_0^{(n)}=(q-1)q^{n-1}$, $[n]_0!=1$ and $[\lambda\langle\phi\rangle_\lambda-1]^{(n)}\to(-1)^n\langle\phi\rangle$, eq. (21) is reduced to an identity; here, one

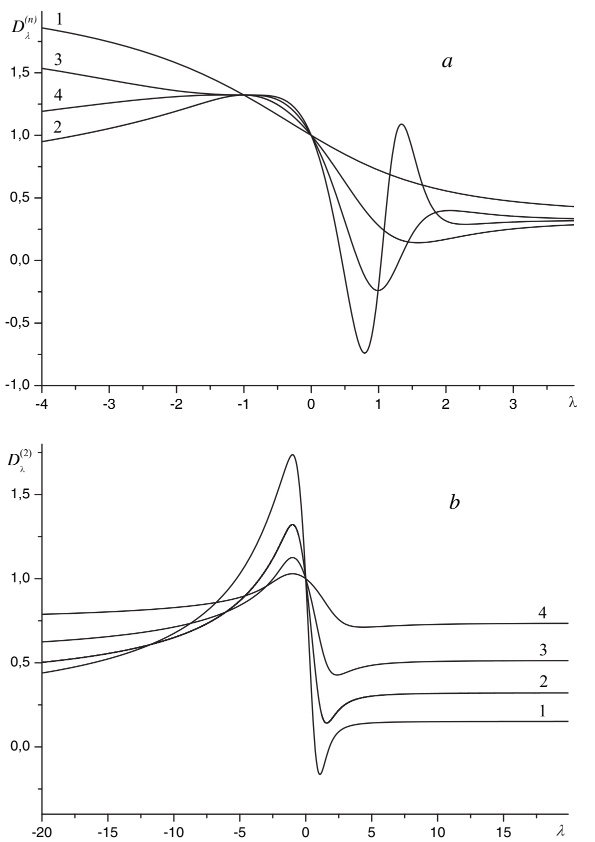

Fig. 1: Fractal dimension coefficients (18) *vs.* deformation of the Cantor binomial set at $p=0.2$ (a) and $n=2$ (b) (curves 1–4 correspond to $n=1,2,3,4$ on the upper panel and $p=0.1,0.2,0.3,0.4$ on the lower one).

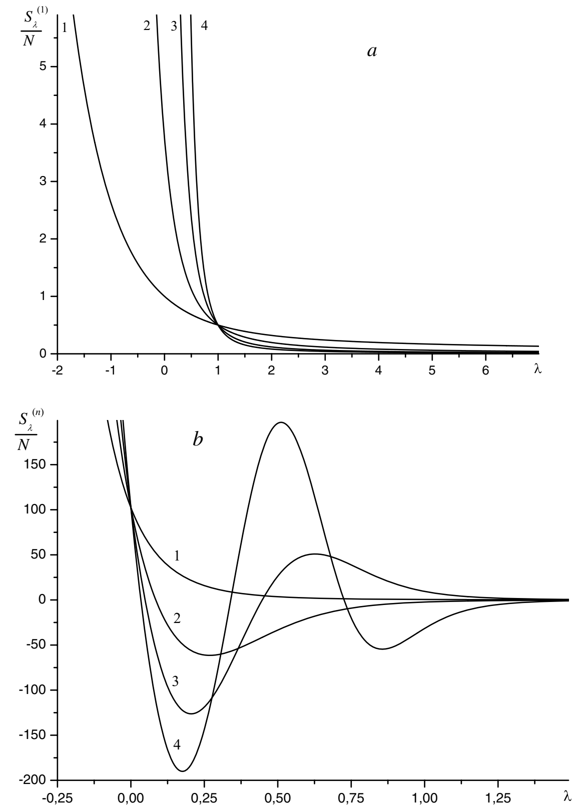

Fig. 2: Effective entropies (19) of the Cantor binomial set at $p=0.2$ and: a) $n=1$ (curves 1–4 correspond to $N=1,4,8,12$); b) $N=10$ (curves 1–4 correspond to $n=1,2,3,4$).

has the uniform distribution $P_i(q,0)=1/W$ for an arbitrary p_i and the average is $\langle\phi\rangle_q^0=W^{-1}\sum_i\phi_i$. At $\lambda=1$, when $[\langle\phi\rangle_1-1]^{(n)}=0$, eq. (21) yields the ordinary average $\langle\phi\rangle=\sum_i\phi_i p_i$ because the distribution $P_i(q,1)$ is reduced to p_i. Setting $\sum_{n=1}^{\infty}(-1)^{n-1}\langle\phi\rangle_{\lambda^n}\to\langle\phi\rangle_\infty$ for $\lambda\gg1$, where $(q-1)_\lambda^{(n)}\sim(-1)^{n-1}(q-1)\lambda^{n(n-1)/2}$, $[n]_\lambda!\sim\lambda^{n(n-1)/2}$ and $[\lambda\langle\phi\rangle_\lambda-1]^{(n)}\sim\lambda^n\langle\phi\rangle_{\lambda^n}$, one obtains the simple dependence $\lambda\langle\phi\rangle_q^\lambda=\langle\phi\rangle+(q-1)\langle\phi\rangle_\infty$, according to which the average $\langle\phi\rangle_q=\langle\phi\rangle_q^q$ tends to the limit $\langle\phi\rangle_\infty$ with the growth of q.

Examples. – To demonstrate the approach developed, we consider initially the simplest example of the Cantor binomial set [3]. It is generated by the N-fold division of the unit segment into equal parts with elementary lengths $\ell=(1/2)^N$, then each of these is associated with the binomially distributed products $p^m(1-p)^{N-m}$, $m=0,1,\dots,N$ of probabilities p and $1-p$. In such a case, the partition function (2) takes the form $Z_q=[p^q+(1-p)^q]^N$ to be equal to $Z_q=\ell^{\tau_q}$ with the mass exponent $\tau_q=\frac{\ln[p^q+(1-p)^q]}{\ln(1/2)}$ [3]. Related dependences of the fractal dimension coefficients (18) on the deformation parameter are depicted in fig. 1 for different orders n and probabilities p. As the upper panel shows, at the given p the monotonic decay of the fractal dimension $D_\lambda^{(n)}$, being usual at $n=1$, transforms into the non-monotonic dependences, whose oscillations are the stronger the higher is the order n. According to the lower panel, such behavior is kept with variation of the probability p, whose growth narrows the dimension spectrum. In contrast to the fractal dimensions (18), the entropy coefficients (19) depend on the effective number of particles N. This dependence is demonstrated by means of the Tsallis entropy $S_\lambda=\mathcal{S}_\lambda^{(1)}$ depicted in fig. 2a: with the deformation growth, this entropy decays the faster the higher is N (by this, the specific value S_λ/N remains constant at $\lambda=1$). According to fig. 2b, with increase of the order n, the monotonically decaying dependence $\mathcal{S}_\lambda^{(1)}$ transforms into the non-monotonic one, $\mathcal{S}_\lambda^{(n)}$, whose oscillations grow with n. Typically, for arbitrary values of n and p, the magnitude $\mathcal{S}_0^{(n)}$ (being equal to 2^N for the Cantor binomial set) remains constant.

As a second example we consider the time series of the currency exchange of euro to US dollar in the course of the years 2007–2009 which include the financial crisis (data are taken from the website `www.fxeuroclub.ru`). To ascertain the crisis effect, we study the time series intervals before (January, 2007–May, 2008) and after (June, 2008–October, 2009) the crisis[1]. Moreover, we restrict

[1]The point of the crisis is fixed at the condition of maximal value of the time series dispersion.

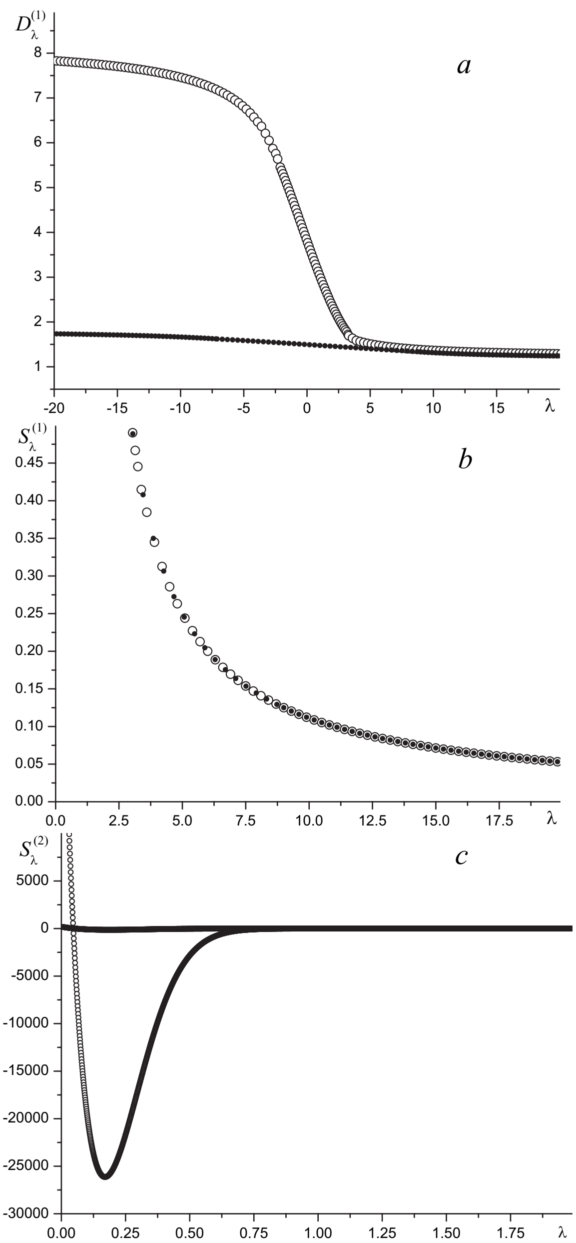

Fig. 3: a) Fractal dimension coefficients (18) at $n=1$ for the time series of the currency exchange of euro to US dollar; related entropy coefficients (19) at $n=1$ (b) and $n=2$ (c) (bullets correspond to the time interval before the financial crisis, circles to that after the crisis).

ourselves to considering the coefficients (18) and (19) of the lowest orders n which make it possible to visualize the difference between the fractal characteristics of the time series intervals pointed out. Along this way, we base on the method of the multifractal detrended fluctuation analysis [11] to find the mass exponent $\tau(q)$, whose use yields the dependences depicted in fig. 3. Comparison of the data taken before and after the financial crisis shows that it affects already the fractal dimension coefficient of the lowest order $n=1$, but has no effect on the Tsallis entropy related to $n=1$, while the entropy coefficient

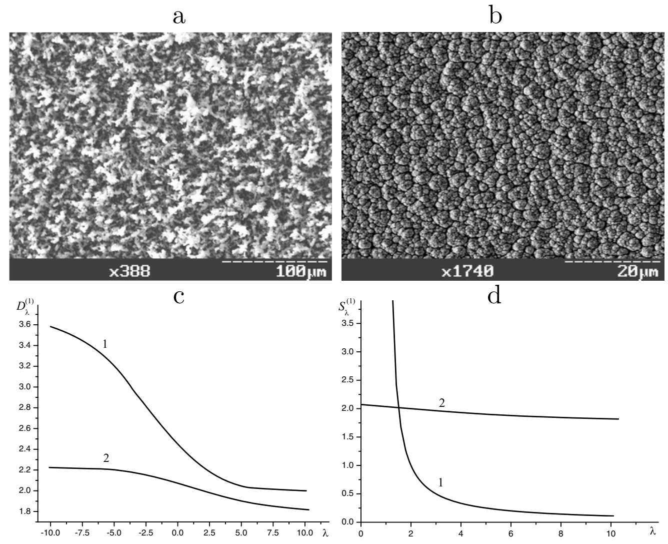

Fig. 4: Scanning electron microscopy images of *ex situ* grown carbon (a) and titanium (b) condensates; corresponding fractal dimensions (c) and entropies (d) at $n=1$ (curves 1, 2 relate to carbon and titanium, respectively).

of the second order $n=2$ is found to be strongly sensible to the crisis. Thus, this example demonstrates visually that the generalized multifractal characteristics elaborated on the basis of the developed formalism allow one to study the subtle details of self-similarly evolving processes.

The last example is concerned with the macrostructure of condensates which have been obtained as a result of sputtering of substances in accumulative ion-plasma devices [12]. The peculiarity of such a process is that its use has allowed to obtain porous condensates of the type shown in figs. 4a and b for carbon and titanium, respectively. These condensates are seen to have apparent fractal macrostructure, whose handling gives the fractal dimension spectra and the entropies depicted in figs. 4c and d, respectively. Comparing these dependences, we can convince the reader that the difference between the carbon condensate, which has a strongly rugged surface, and the titanium one, becomes apparent already by using the fractal dimension and entropy coefficients (18), (19) related to the lowest order $n=1$. Thus, in the case of multifractal objects with strongly different structures the use of the usual multifractal characteristics appears to be sufficient.

Conclusion. – Generalizing the multifractal theory [5], we have represented the partition function (3), the mass exponent (15), and the average $\langle\phi\rangle_q^\lambda$ of the self-similarly distributed random variable as deformed series in powers of the difference $q-1$. Coefficients of these expansions are shown to be determined by the functional binomial (8) that expresses both multifractal dimension spectra (18) and generalized Tsallis entropies (19) related to manifold deformations. We have found eq. (21) for the average related to the generalized probability (20) subject to the deformation. Recently, making use of above formalism has allowed us to develop a field theory for self-similar statistical systems [13].

As examples of multifractal sets in mathematical physics, objects of solid-state physics, and processes in econophysics, we have applied the formalism developed to the analysis of Cantor manifolds, porous-surface condensates, and exchange currency series, respectively. The study of the Cantor set has shown that both fractal dimension coefficients (18) and entropies (19) coincide with the usual multifractal characteristics in the lowest order $n = 1$, but display a very complicated behaviour at $n > 1$. On the contrary, the consideration of both carbon and titanium surface condensates has shown that their macrostructures can be characterized by the use of the usual fractal dimension and entropy coefficients related to the order $n = 1$. A much more complicated situation takes place in the case of the time series type of currency exchange series. Here, a difference between the various series is displayed for the fractal dimension coefficients already in the lowest order $n = 1$, but the entropy coefficients coincide at $n = 1$ and become different at $n = 2$. This example demonstrates the need to use the generalized multifractal characteristics obtained within the framework of the formalism developed.

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